Geometric structure of Data through Deep Learning models

Rita Fioresi, FaBiT, Unibo

May 2, 2023

Plan of the Talk

- Introduction to Deep Learning

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- Information Geometry

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- Data manifold and dimensionality reduction

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- Main results (joint work with Grementieri)

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- Main results (joint work with Grementieri)
- Conclusions and future directions

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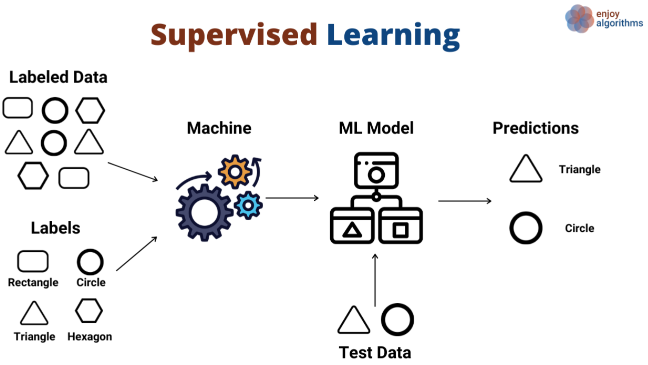
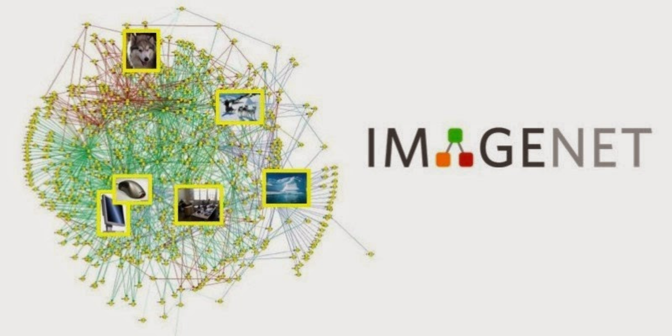


Imagenet Challenge ILSVRC: ImageNet Large Scale Visual Recognition Challenge



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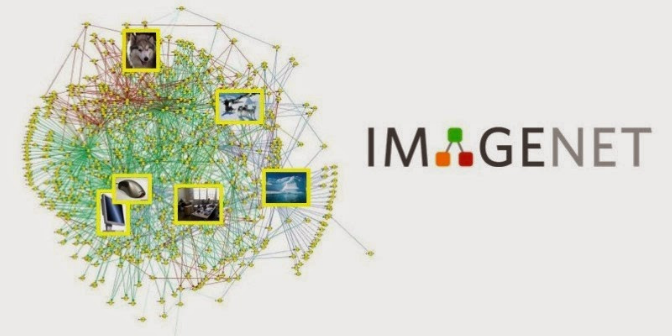

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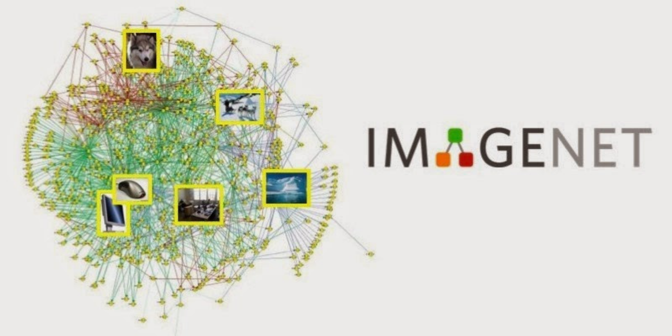

- **2010** 20000 images, 20 categories, 25% error.
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- **2015** 1 million images, 1000 categories: 4% error.

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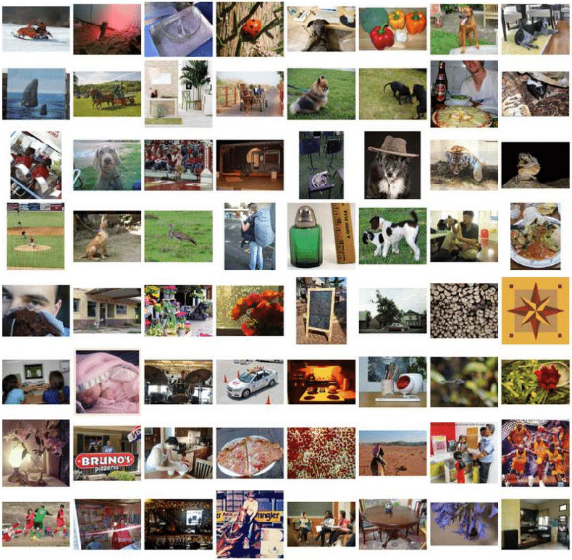
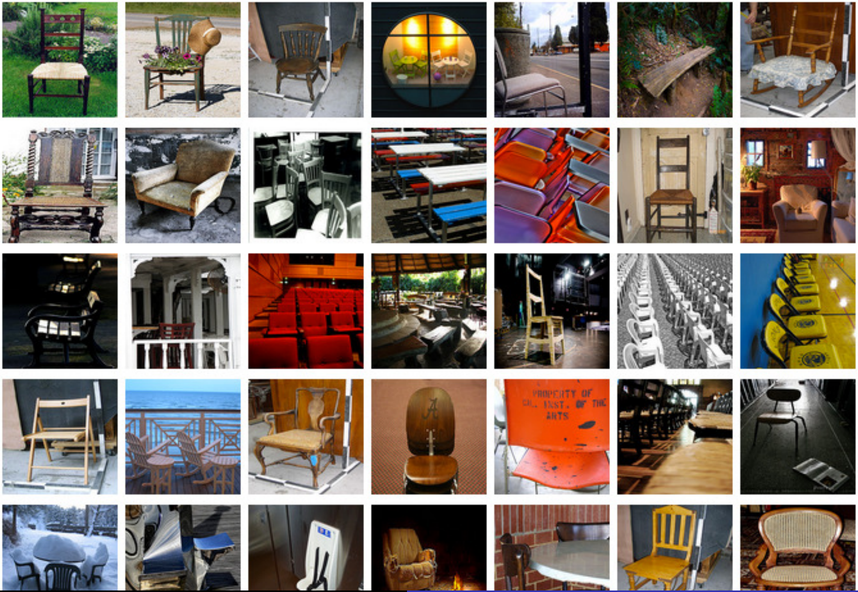
Images in Imagenet



Images in Imagenet category "chair"



Benchmark datasets: MNIST and CIFAR10

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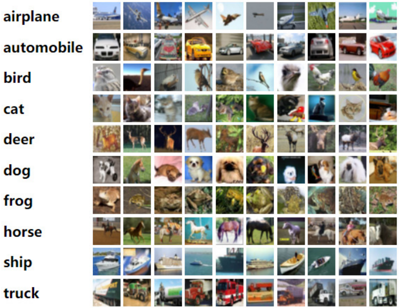

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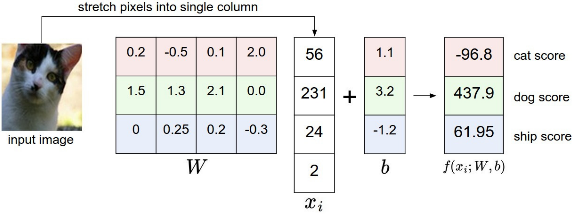

Ingredients for Deep Learning

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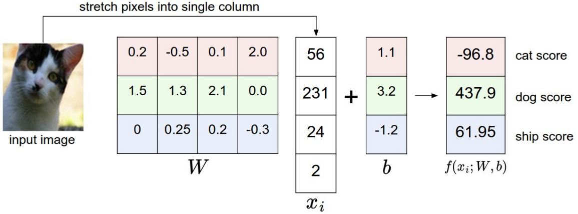

- **Loss function**: measures error
  (L_i datum i loss, y_i correct label)

$$L_i = -log \frac{e^{f_{y_i}}}{\sum_j e^{f_j}} = -f_{y_i} + log \sum_j e^{f_j}, \qquad L = \sum_i L_i$$

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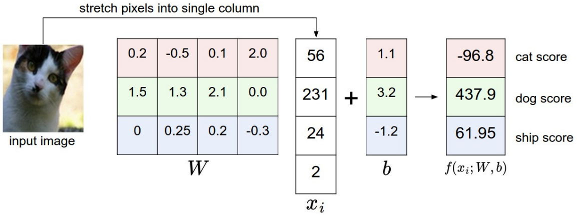

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- **Optimizer**: for weights update "minimizes" the Loss

$$w_{ij}(t+1) = w_{ij}(t) - \alpha \nabla L_{\mathrm{stoc}}, \qquad \nabla L_{\mathrm{stoc}} = \sum_{i=1}^{32} \nabla L_{\mathrm{rand(i)}}$$

Training

Divide the dataset (ex. CIFAR10):
80% Data for **training**
10% Data for **validation**
10% Data for **test** (ONCE)

1. **Learning**: determine weights **parameters**

Accuracy: percentage of accurate predictions on tests set.

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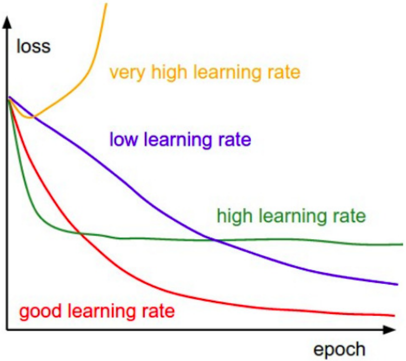
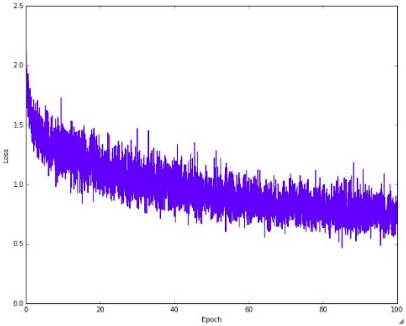
Epoch$=$ $\|$Training set$\|/\|$minibatch size$\|$.
NOTE: measure accuracy every 10-20 epochs.
Example: 40000 training set (CIFAR10), 32 images in minibatch, 1 epoch$=40000/32$ updates.

Loss accuracy in epochs: CIFAR10





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- e.g. $\mathcal{B} = 8, 16, 32$

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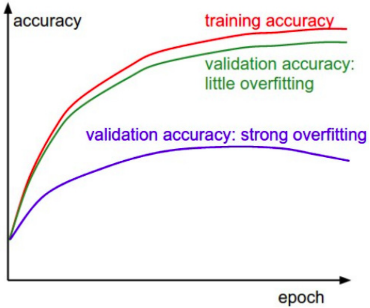
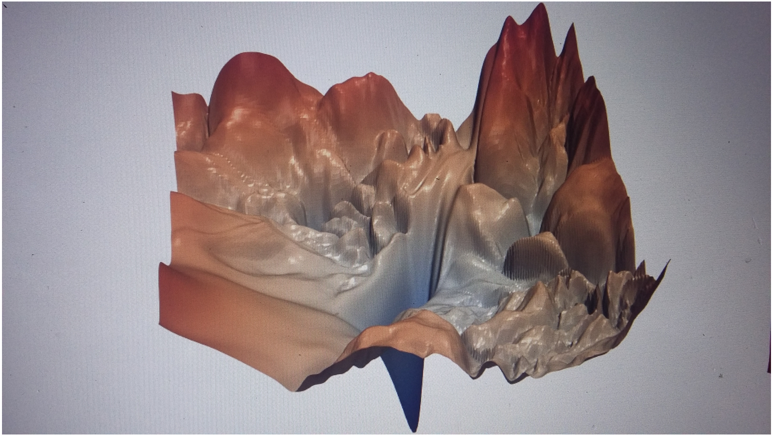
Test

ATTENTION!: use test set ONCE to avoid **overfitting**!



Validation technique: cross validation=rotation of the training set.

Loss Landscape



Loss (projection) as function of weights.

Information Geometry

Information Geometry: studies geometrical structures on manifolds in the parameter space and the data domain.

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The Kullback-Leibler divergence measures the “difference” between the two probability distributions the “empirical distribution” p and the “true distribution” q.

LOSS: Softmax S and Cross Entropy Loss L

$L(x, w) = -\log[S(x, w)] = -\log[e^{s_{y_j}(x)}/(e^{s_1(x)} + \cdots + e^{s_N(x)})]$
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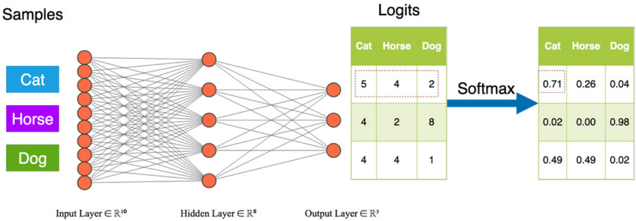

$$S(x) = \frac{e^5}{e^5 + e^4 + e^2} = 0.71$$

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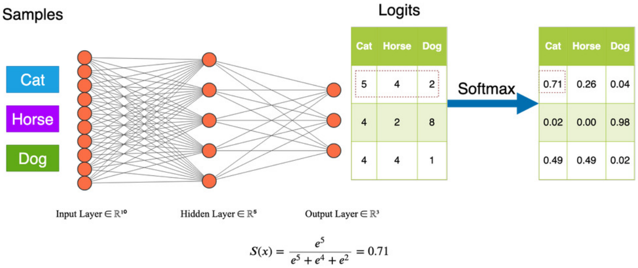

$$Loss = -\log(S_{cat}) - \log(S_{horse}) - \log(S_{dog}) =$$

$$= -\log(0.71) - \log(0.002) - \log(0.02) = 0.34 + 6 + 3.91 = 10.25$$

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The Local Data matrix G provides a **natural metric on the data domain**.

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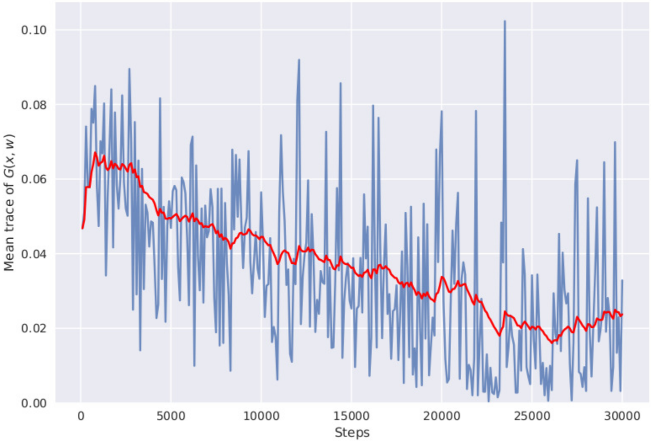

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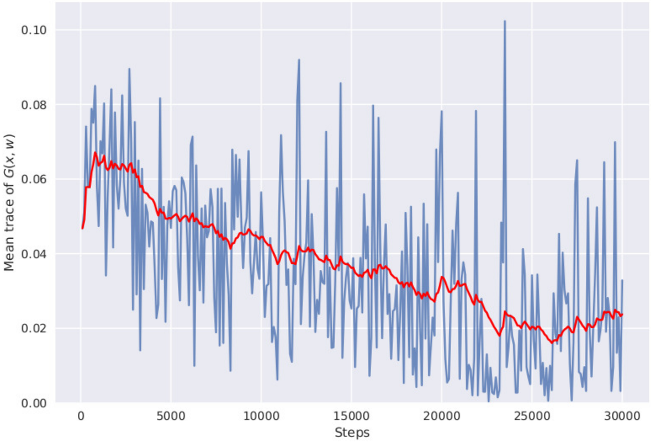

This is why we do not want a fully trained model: the information is lost at equilibrium!

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Dataset	$G(x, w)$ size	$\mathrm{rank}\ G(x, w)$ bound
MNIST	784	10
CIFAR-10	3072	10
CIFAR-100	3072	100
ImageNet	150528	1000

C: is the number of classes for our classification task

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2. The dimension d of every submanifold (every leaf of the foliation) is bounded by the number of classes C of our classification model: $d << n$ (e.g. MNIST $d = 9 << 784$).

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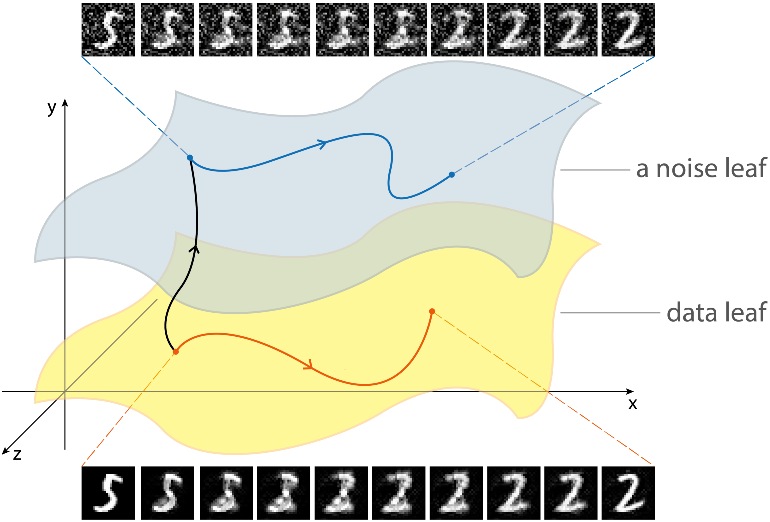

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Remark: This is not true for the distribution via the Fisher matrix!

$$w \mapsto \mathcal{D}'_w := (\ker F(w))^\perp$$

is **not** involutive (e.g. MNIST, lenet).

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- Not all points on the data leaf are in the data set, but they represent *symbols*.

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Moving around in **on the data leaf**:

- We can connect any two data=images.

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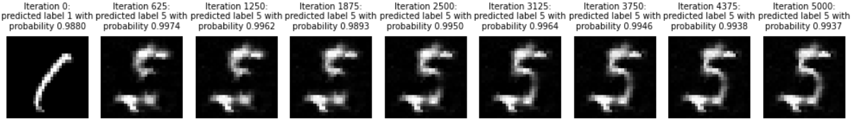

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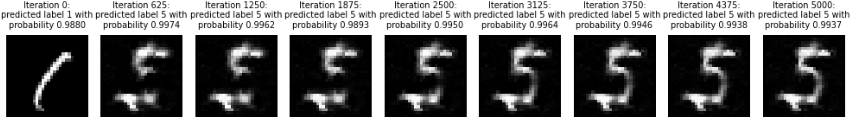

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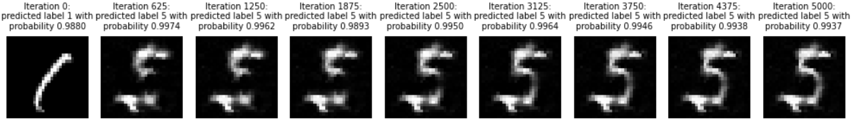

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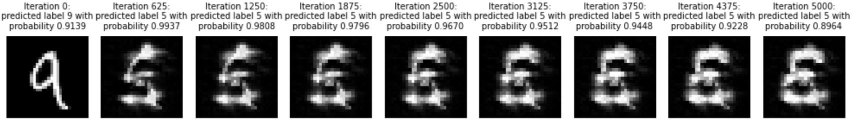

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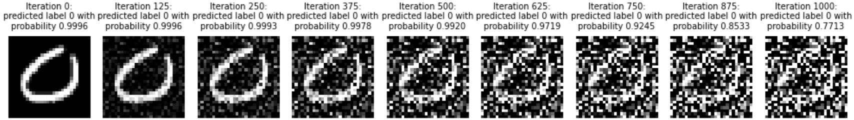

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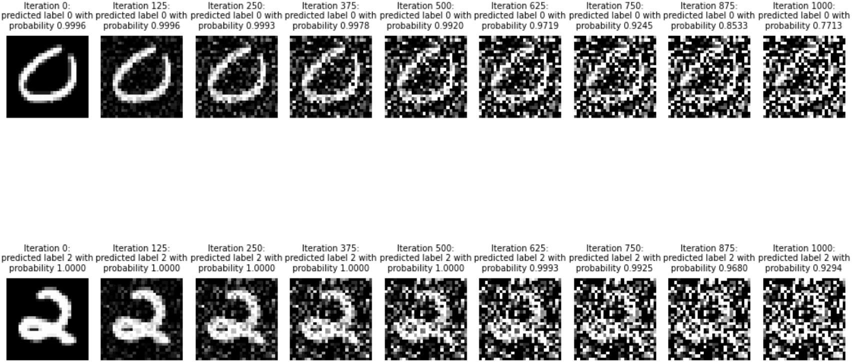

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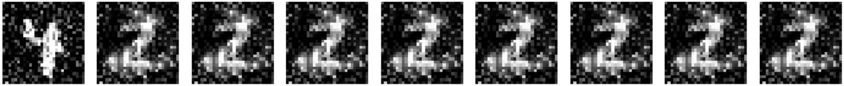

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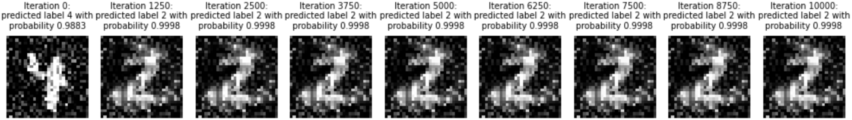
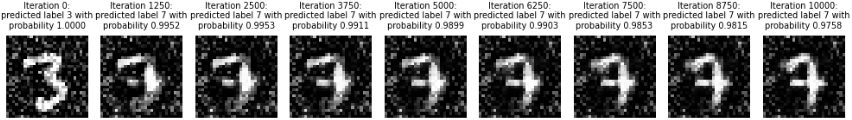
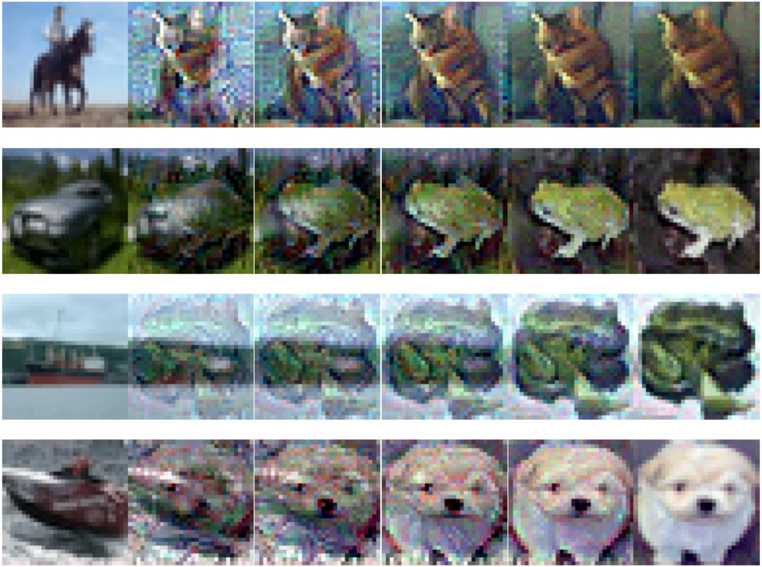




Moving on the data manifold: CIFAR10



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- Navigating the data leaf can lead to data augmentation and efficient denoising algorithms

Bibliography

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