Spectral problems in integrable systems, Grassmannians and graph theory

Simonetta Abenda (UniBo)

Caligola Workshop 'A quantum day in Bologna'

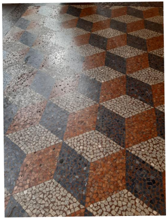
June 9, 2023

CaLIGOLA

Cartan geometry, Lie and representation theory, Integrable Systems, quantum Groups and quantum computing towards the understanding of the geometry of deep Learning and its Applications

Marie Skłodowska-Curie Actions

Summary



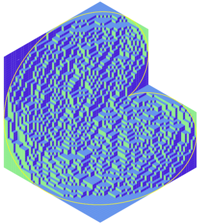
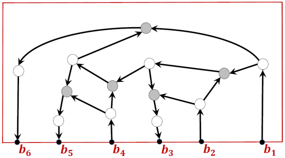


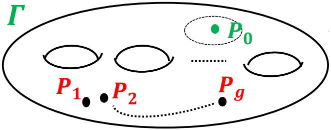
Figure 1.

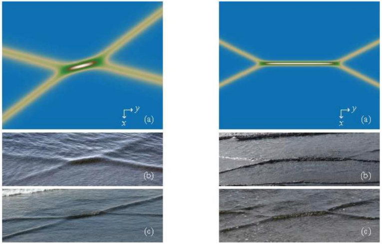
Left:https://en.wikipedia.org/wiki/Kadomtsev%E2%80%93Petviashvili_equation
Center: Sidewalk at the entrance of St Mary Magdalene's Church (Bologna, Via Zamboni)
Right:R. Kenyon, A. Okounkov, Notices AMS 2005

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 9 & 5 \end{pmatrix}$$



Wikipedia





M.A. Ablowitz and D.E. Baldwin Phys. Rev E , v. 86 (2012)

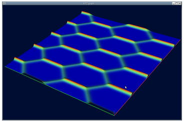
$$\text{KP} - 2\ \text{equation}\ \ [\text{KP} - 1970]:\ \ (-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

is the first member of the most relevant $2+1$ integrable hierarchy [ZS-1974].

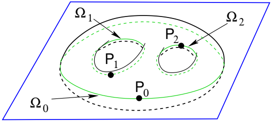
Krichever [Kr-1976] characterized KP finite–gap solutions algebraic-geometrically.

Spectral problem for real regular KP-2 finite–gap solutions

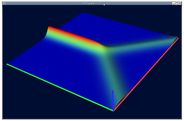
Real regular finite–gap KP-2 solutions ↔ spectral data on M–curves



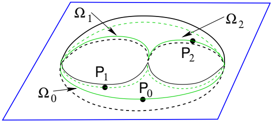
↔ [DN-1988]



Solitonic ↓ limit [AG20..] ↑ ↑



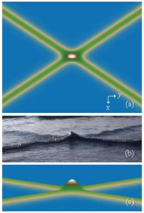
→ [AG-2018a]

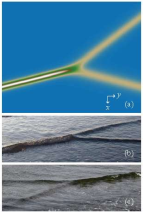


Real regular multiline KP-2 solitons → spectral data on reducible spectral curve [AG-2018a]

[CK-2009],[KW-2014]: Use combinatorial structure of $Gr^{\mathrm{TNN}}(k, n)$ to characterize the asymptotic behaviour and tropical limit of Real regular multiline KP-2 solitons

[AG-2018a,AG-2019,AG-2022c]: Use combinatorial structure of $Gr^{\mathrm{TNN}}(k, n)$ to get the spectral data of real regular finite–gap KP-2 solutions solving a degenerate spectral problem for real regular multiline KP solitons





Left: $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{7} \end{bmatrix}$ $k_1 := \frac{-7}{12}, k_2 := \frac{-1}{12}, k_3 := \frac{1}{12}, k_4 := \frac{7}{12}$

Right: $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}, k_1 := 0, k_2 := \frac{1}{2}, k_3 := 1$

Wronskian method [Mat-1979], [FN-1983], [Mal-1991]:

$$f^{(i)}(x,y,t) = \sum_{i=1}^{n} A^i_j \exp(\kappa_j x + \kappa_j^2 y + \kappa_j^3 t), \quad i \in [k]$$

$$\tau(x,y,t) = \mathrm{Wr}_x(f^{(1)}, \dots, f^{(k)}) = \sum_{1\le j_1 < \cdots < j_k \le n} \Delta_{[j_1,\dots,j_k]}(A) E_{j_1,\dots,j_k}(x,y,t)$$

KP-2 soliton solution: $u(x,y,t) = 2\partial_x^2 \log(\tau(x,y,t))$

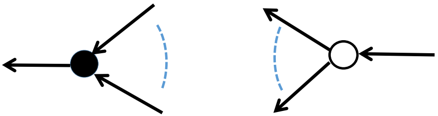
- same $u(x,y,t)$ if recombine rows of $A \implies [A] \in Gr(k,n) = GL_{\mathbb{R}}(k)\backslash Mat_{\mathbb{R}}(k,n)$
- u is bounded for real $(x,y,t) \iff [A] \in Gr^{\mathrm{TNN}}(k,n) = GL^+_{\mathbb{R}}(k)\backslash Mat^{\mathrm{TNN}}_{\mathbb{R}}(k,n)$ [KW-2013])

◇ [Lusz-1990s] generalizes the classical notion of total positivity in GL_n to reductive Lie groups and generalized partial flag varietes; cell decomposition of $(G/P)_{\geq 0}$ (Rietsch, Ph.D. thesis).

◇ [Pos-2006] characterizes the cell decomposition of $Gr^{\text{TNN}}(k,n)$ combinatorially and using graph theory:

A positroid cell in $Gr^{\text{TNN}}(k,n)$ is represented by an equivalence class of perfectly orientable planar bicolored graphs in the disk (real positive weights on edges of the graph):

- n univalent vertices on the boundary of the disk and k of them are sources in each perfect orientation;
- At each internal black vertex, exactly one edge oriented outward;
- At each internal white vertex, exactly one edge oriented inward.



▷ Multi-line soliton solutions are represented by points $[A] \in Gr^{\mathrm{TNN}}(k,n)$

▷ Direct spectral problem [Mal-1991]: $(\Gamma_0 = CP^1, P_0)$ and $P_1, \dots, P_k \in [\kappa_1, \kappa_n]$

▷ Inverse spectral problem [AG-2019, AG-2022c]: Postnikov's graphs $\mathcal{G}$ representing $[A]$ are dual to reducible spectral curves Γ for KP-2 soliton solutions

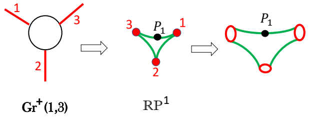
boundary of disk $\leftrightarrow \Gamma_0$, vertices, $\mathbb{CP}^1$, edges $\leftrightarrow$ double points, faces $\leftrightarrow$ ovals

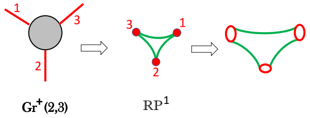
• Construct and solve a system of relations on $\mathcal{G}$ [AG-2022a,AG-2022b] $\implies$ value of KP-2 wave–function at double points and one divisor point in each finite oval [AG-2022c]

◇ Total non–negativity $\implies$ reality and regularity DN conditions [AG-2018a]

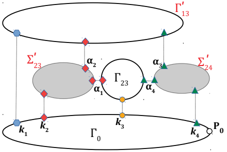
◇ Kasteleyn theorem rules the system of relations if the graphs is bipartite [A-2021] ← Connection with dimer model!!!

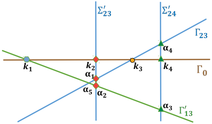
◇ The construction may be interpreted as amalgamation of little positive Grassmannians [AG-2022a,AG-2022b] ← Connection with cluster integrable systems [GK-2013], [GSV-2010] and with the amplituhedron [ABCGPT-2016]

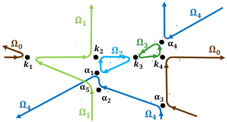




Soliton lattices of KP-2 and desingularization of spectral curves in $Gr^{\mathrm{TP}}(2,4)$ [AG-2018b]





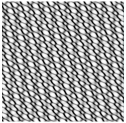
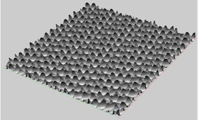
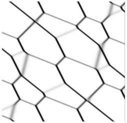
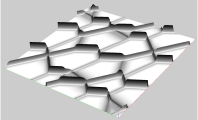


$$0 = P_0(\lambda,\mu) = \mu \cdot (\mu - (\lambda - \kappa_1)) \cdot (\mu + (\lambda - \kappa_2)) \cdot (\mu - (\lambda - \kappa_3)) \cdot (\mu + (\lambda - \kappa_4)).$$

Genus 4 M–curve after desingularization:

$$\Gamma(\varepsilon) \ : \qquad P(\lambda,\mu) = P_0(\lambda,\mu) + \varepsilon(\beta^2 - \mu^2) = 0, \qquad 0 < \varepsilon \ll 1,$$

$\beta = \frac{\kappa_4 - \kappa_1}{4} + \frac{1}{4}\max\{\kappa_2 - \kappa_1, \kappa_3 - \kappa_2, \kappa_4 - \kappa_3\}$,
$\kappa_1 = -1.5, \quad \kappa_2 = -0.75, \quad \kappa_3 = 0.5, \quad \kappa_4 = 2.$

Level plots for KP-II finite gap solutions: $\epsilon = 10^{-2}$ [left], $\epsilon = 10^{-18}$ [right].
Horizontal axis is $-60 \le x \le 60$, vertical axis is $0 \le y \le 120$, $t = 0$.
White (black) = lowest (highest) value of u.

Dimer models on surface graphs (with boundaries)

• Dimer models were introduced in [Kas-1961] and [TF-1961] to describe crystal surfaces at equilibrium like partially dissolved salt crystals.

• States in dimer models are perfect matchings between vertices of the graph where only adjacent vertices are matched. If the surface has boundaries, all internal vertices have to be matched.

• The probability of a state is the product of the edge weights of the dimer configuration.

• The partition function can be written as a linear combination of N Pfaffians of $m \times m$ Kasteleyn matrices, where

- m = number of vertices
- N = number of non equivalent Kasteleyn orientations of the surface graph. N depends on genus g and boundary components of the surface where the graph is embedded. [Kas-1961], [GL-1999], [Tes-2000], [CR-2008]

• $g = 1 \quad \Longrightarrow \quad N = 4$

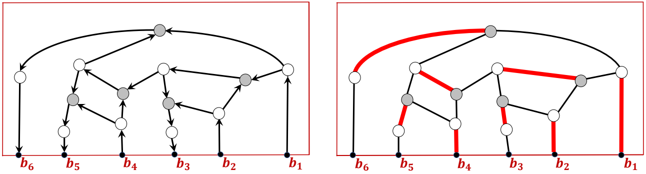
• The most studied case is $g = 1$: [GK-2013] associate quantum integrable systems to dimer models on bipartite graphs on a torus in such a way that the positive part of the phase space coincides with the assignment of algebraic geometric data on the ovals of the Harnack curves associated to such models in [KO-2006]

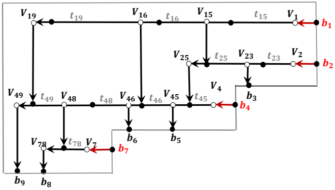
[PSW-2009]: Dimer configuration on $\mathcal{G} = (\mathcal{V} = \mathcal{B} \cup \mathcal{W}, \mathcal{E})$ is a collection M of edges of $\mathcal{G}$ that contains exactly once internal vertices, and at most once the n boundary vertices.

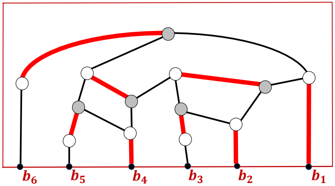
$$k = \partial M = \{i \in [n] \ : \ \text{black boundary vertex } b_i \in M\} \cup \{i \in [n] \ : \ \text{white boundary vertex } b_i \notin M\}.$$

Perfect orientations $\iff$ dimer configurations

Example: $Gr^{TP}(3, 6)$:







[Pos-2006]: $[A] \in Gr^{\mathrm{TNN}}(k,n)$:

$$A^r_j = (-1)^{\sigma_{i_r j}} \sum_{P: b_{i_r} \mapsto b_j} (-1)^{\mathrm{Wind}(P)} wt(P)$$

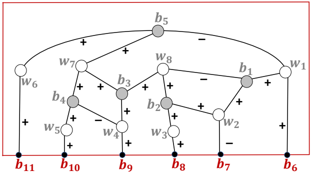
$\sigma_{i_r j} = \#\{$ sources between i_r and $j\}$;

[Lam-2016]: Weight of dimer state M:

$$wt(M) = \prod_{e \in M} wt(e)$$

The partition function $Z(G, wt; \partial M)$ relative to $\partial M = I$ is the I-th Plücker coordinate of $[A] \in Gr^{\mathrm{TNN}}(k,n)$:

$$Z(G, wt; \partial M) = \sum_{M\,:\,\partial M = I} wt(M) = D_I(A)$$



[AG-2022a,AG-2022b]: To each planar bicolored graph in the disk, we associate a system of relations ruled by a unique equivalence class of signatures σ which gives Postnikov boundary measurement map and solve it at internal vertices generalizing [Tal-2008].

[A-2021]: For bipartite graphs in the disk σ is the Kasteleyn signature characterized topologically in [Sp-2016]: maximal minors $\det(K^{wt})_I$ of the Kasteleyn matrix K^{wt}

$$(K^{wt})_b^w = \begin{cases} \sigma_{b,w} wt_{b,w}, & \text{if } (b,w) \text{ is an edge;} \\ 0, & \text{otherwise,} \end{cases}$$

are the dimer partition functions $Z(G, wt; \partial M)$ for $\partial M = I$, that is the Plücker coordinates $D_I(A)$ of $[A] \in Gr^{\mathrm{TNN}}(k,n)$ represented by the given network:

$$\det(K^{wt})_I = Z(G, wt; \partial M) = \sum_{M \,:\, \partial M = I} wt(M) = D_I(A)$$

$$K^{wt} \mapsto \begin{array}{c} \\ N \\ k \end{array} \begin{array}{cc} N & n \\ \left(\begin{array}{c|c} \mathrm{Id}_N & * \\ \hline 0 & A \end{array}\right) \end{array}$$

Kasteleyn system of relations and KP wave function

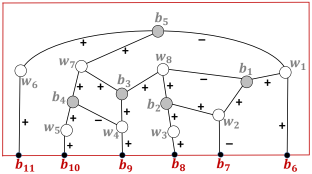
- K^{wt} Kasteleyn matrix $\qquad$ V a vector space

Kasteleyn system of relations $(v^{(k)} = \{v_b^{(k)} : b \in \mathcal{B}\}, R_w)$:

▷ $v_b^{(k)}$ is an element in V assigned to the black vertex $b \in \mathcal{B}$;

▷ At white vertex $w \in \mathcal{W}$: $R_w(v^{(k)}) \equiv \sum_{b\in\mathcal{B}} (K^{wt})_b^w \, v_b^{(k)} \equiv \sum_{b\in\mathcal{B}} \sigma_{bw} \, wt_{bw} \, v_b^{(k)} = 0.$

◇ [A-2021]: KP soliton wave function on Γ_0: $0 \equiv \mathfrak{D} f_i(\vec{x}) \equiv \sum_{j=1}^n A_j^i \, \psi(\kappa_j, x, y, t)$



Assign at boundary vertex b_j: $v_{b_j}^{(k)} = \psi(\kappa_j, x, y, t)$

Solve the system and get $\psi(\kappa, x, y, t)$

at the double points of the reducible curve!

◇ [AG-2022a], [AG-2022b]: explicit solution to the system of relations $\Longrightarrow$

[AG-2022c]: explicit computation of the KP wave - function at double points and of the KP divisor.

Open problems

• Classification of $\mathtt{M}$–curves associated to Postnikov plabic graphs ([A-2017]: Le-graph for $Gr^+(1,n)$ are hyperelliptic genus $n-1$; [AG-2018b]: $Gr^+(2,4)$ is trigonal genus 4)

• Identify varieties in positroid cells associated to reductions of KP-hierarchy (KdV, Boussinesq, ...)

• Understand connection with Kodama-Williams classification of asymptotic behavior of KP-solitons and tropical limit

• Use systems of relations to solve other problems in integrable systems/statistical mechanics/theoretical physics for surface graphs with boundaries.

[A-2017] S. Abenda *On a family of KP multi–line solitons associated to rational degenerations of real hyperelliptic curves and to the finite non–periodic Toda hierarchy*, J.Geom.Phys. **119** (2017) 112–138.

[A-2021] S. Abenda, *Kasteleyn theorem, geometric signatures and KP-II divisors on planar bipartite networks in the disk*, Math. Phys. Anal. Geom. **24**, Art. #35, (2021): 64 pp.

[AG-2018a] S. Abenda, P.G. Grinevich, *Rational degenerations of M-curves, totally positive Grassmannians and KP–solitons*, Commun. Math. Phys. **361** Issue 3 (2018) 1029–1081.

[AG-2018b] S. Abenda, P.G. Grinevich *Real soliton lattices of the Kadomtsev-Petviashvili II equation and desingularization of spectral curves corresponding to* $Gr^{TP}(2,4)$. Proc. Steklov Inst. Math. **302** (2018), no. 1, 1–15.

[AG3-2019] S. Abenda, P.G. Grinevich, *Reducible M-curves for Le-networks in the totally-nonnegative Grassmannian and KP–II multiline solitons*, Sel. Math. New Ser. **25**, no. 3 (2019) 25:43.

[AG-2022a] S. Abenda, P.G. Grinevich, *Edge vectors on plabic networks in the disk and amalgamation of totally non-negative Grassmannians*, Adv. in Mathem. **406**, Art. no. 108523, 57 pp. (2022).

[AG-2022b] S. Abenda, P.G. Grinevich, *Geometric nature of relations on plabic graphs and totally non-negative Grassmannians*, IMRN **2022**, Art. no. rnac162, 66 pp. (2022)

[AG-2022c] S. Abenda, P.G. Grinevich, *Real regular KP divisors on M-curves and totally non-negative Grassmannians*, LMP, **112**, Art. no: 115 , pp. 1 - 64 (2022).

[ABCGPT-2016] N. Arkani–Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov, and J. Trnka, *Grassmannian geometry of scattering amplitudes.* Cambridge University Press, Cambridge, 2016.

[CK-2009] S. Chakravarty, Y. Kodama, *Soliton solutions of the KP equation and application to shallow water waves.* Stud. Appl. Math. **123** (2009) 83–151.

[CR-2008] D. Cimasoni, N. Reshetikhin, *Dimers on surface graphs and spin structures. II* Commun. Mathem. Phys. **281**, no. 2, (2008), 445-468

[DN-1988] B. A.Dubrovin, S.M. Natanzon, *Real theta-function solutions of the Kadomtsev-Petviashvili equation.* Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988) 267–286.

[FN-1983] N.C. Freeman, J.J.C. Nimmo *Soliton solutions of the Korteweg de Vries and the Kadomtsev-Petviashvili equations: the Wronskian technique*, Proc. R. Soc. Lond. A **389** (1983), 319–329

[GL-1999] A. Galluccio, M. Loebl, On the theory of Pfaffian orientations I. Perfect matchings and permanents, Electron. J. Combin. **6**, res. paper 6 (1999),18 pp.

[GSV-2010] M. Gekhtman, M. Shapiro, and A. Vainshtein, *Cluster algebras and Poisson geometry.* Mathematical Surveys and Monographs, 167. American Mathematical Society, Providence, RI, (2010), xvi+246 pp.

[GK-2013] A.B. Goncharov, R. Kenyon, *Dimers and cluster integrable systems*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 5, 747–813.

[KP-1970] B.B. Kadomtsev, V.I. Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15** (1970) 539-541.

[Kas-1961] P.W. Kasteleyn, *The statistics of dimers on a lattice.I. The number of dimer arrangements on a quadratics lattice*, Physica **27** (1961), 1209–1225.

[KO-2006] R. Kenyon and A. Okounkov, *Planar dimers and Harnack curves*, Duke Math. J. **131**, no. 3 (2006): 499–524.

[KW-2013] Y. Kodama, L.K. Williams, *The Deodhar decomposition of the Grassmannian and the regularity of KP solitons.* Adv. Math. **244** (2013) 979-1032.

[KW-2014] Y. Kodama, L.K. Williams, *KP solitons and total positivity for the Grassmannian.* Invent. Math. **198** (2014) 637-699.

[Kr-1976] I.M. Krichever, *An algebraic-geometric construction of the Zakharov-Shabat equations and their periodic solutions.* (Russian) Dokl. Akad. Nauk SSSR **227** (1976) 291-294.

[Lam-2016] T. Lam, *Totally nonnegative Grassmannian and Grassmann polytopes*, Current developments in mathematics 2014, 51–152, Int. Press, Somerville, MA, (2016).

[Lus-1994] G. Lusztig, *Total positivity in reductive groups. Lie theory and geometry*, Progr. Math. **123**, Birkhäuser Boston, Boston, MA (1994), 531–568.

[Lus-1998] G. Lusztig, "Total positivity in partial flag manifolds." *Representation Theory* 2 (1998), 70–78.

[Mal-1991] T.M. Malanyuk, *A class of exact solutions of the Kadomtsev–Petviashvili equation.* Russian Math. Surveys, **46**:3 (1991), 225–227.

[Mat-1979] V.B. Matveev, *Some comments on the rational solutions of the Zakharov-Schabat equations*. Letters in Mathematical Physics, **3** (1979), 503–512.

[Pos-2006] A. Postnikov, *Total positivity, Grassmannians, and networks.*, arXiv:math/0609764 [math.CO].

[PSW-2009] A. Postnikov, D. Speyer, L. Williams, *Matching polytopes, toric geometry, and the totally non-negative Grassmannian.* J. Algebraic Combin. **30** (2009), no. 2, 173-191.

[Sp-2016] D.E. Speyer,*Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian*, Electron. J. Combin. **23**, no. 2 (2016) Paper 2.24, 7 pp.

[Tal-2008] K. Talaska, *A Formula for Plücker Coordinates Associated with a Planar Network.* IMRN **2008**, (2008), Article ID rnn081, 19 pages.

[TF-1961] H. Temperly, M. Fisher,*The dimer problem in statistical mechanics - an exact result.* Phil. Mag. **6** (1961), 1061–1063.

[Tes-2000] G. Tesler, *Matchings in graphs on non-orientable surfaces*, J. Combin. Theory Ser. B 78(2), 198–231 (2000)

[ZS-1974] V.E. Zakharov, A. B. Shabat, *A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. I*, Funct. Anal. and Its Appl., **8** (1974), Issue 3, 226–235.